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2001 J. Phys. A: Math. Gen. 34 273

(http://iopscience.iop.org/0305-4470/34/2/305)

J. Phys. A: Math. Gen. **34** (2001) 273–279 www.iop.org/Journals/ja PII: S0305-4470(01)15910-X

Gauge covariant observables and phase operators

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Received 28 July 2000, in final form 8 November 2000

Abstract

We characterize those Hilbert space operators which are first moments of the gauge covariant positive operator-valued measures introduced by Lahti and Pellonpää.

We consider the suitability of such measures for describing quantum phase, observing that the gauge angle defining the measure is not always closely related to the spectrum of the first moment operator. We take the position that the first moment operator represents the quantal property the measure is (imperfectly) describing. This is in contrast to the viewpoint that positive operator-valued measures are the fundamental quantum observables.

PACS numbers: 0365, 0230, 0240

1. Introduction

In a recent paper, Lahti and Pelonpää [3] introduce a certain class of positive operator-valued measures (POVMs). In this paper we shall identify and characterize the Hilbert space operators which are the moments of these POVMs, and conversely. The physical meaning of these measures is discussed in [3] and we provide a different point of view in the last part of this paper. Because there is some disagreement about this interpretation, but not the mathematics surrounding it, we shall use the mathematically neutral terminology *gauge positive operator-valued measures*, or GPOVMs, which Lahti and Pelonpää termed *gauge observables*.

The setting of the theory is the Hilbert space $L^2(\mathbb{R})$ (associated with the (irreducible) Schrödinger representation of the canonical commutation relations for one degree of freedom). A distinguished orthonormal basis for this space is the set $\{h_n : n \geqslant 0\}$ of Hermite–Gauss functions (also known as oscillator eigenfunctions), which is, at the same time, a Schauder basis for the space $\mathcal{S}(\mathbb{R})$ of smooth functions which, together with all derivatives, vanish at infinity faster than any polynomial, and a topological basis for its dual $\mathcal{S}'(\mathbb{R})$. Critical to the measures introduced by Lahti and Pelonpää [3, 4] is the strongly continuous unitary group $\{U_\theta : \theta \in \mathbb{T}\}$ of gauge transformations, generated by the self-adjoint number operator $\overline{N} = \sum_{n\geqslant 0} nP_n$, where P_n is the projection onto the one-dimensional subspace spanned by h_n. Here $\mathbb{T}$ is the multiplicative group of complex numbers of modulus 1—however, for simplicity we shall adopt a real parametrization for $\mathbb{T}$, and write $\mathbb{T} = [0, 2\pi)$, with the group action being described as addition modulo 2π. We do not need to specify the domain of the number operator $\overline{N}$, as $\mathcal{S}(\mathbb{R})$ is a core of self-adjointness for it, and all calculations can be performed in that space. We shall, as usual, write N for its restriction there.

With these preliminaries introduced, we can give the definition of the class of POVMS introduced by Lahti and Pelonpää.

Definition 1.1. *By a gauge positive operator-valued measure, or GPOVM, is meant a positive operator-valued measure E from the Borel subsets $\mathcal{B}(\mathbb{T})$ of $\mathbb{T}$ into the contraction operators on $L^2(\mathbb{R})$, which is covariant under the gauge group:*

$$U_\theta E(V) U_\theta^{-1} = E(V + \theta) \qquad V \in \mathcal{B}(\mathbb{T}) \quad \theta \in \mathbb{T}. \tag{1.1}$$

Note that the translation operation in $\mathcal{B}(\mathbb{T})$ also needs to be understood as being calculated modulo 2π. The set of all GPOVMs on $L^2(\mathbb{R})$ will be written $\mathcal{GM}$.

The main result of Lahti and Pelonpää is to characterize GPOVMs in terms of a particular space of doubly indexed one-sided complex sequences, namely the space $\mathfrak{k}$ of all such double sequences $(c_{mn})_{m,n\geqslant 0}$ for which

$$c_{nn} = 1 \qquad n \geqslant 0 \tag{1.2}$$

and such that, for any $k \geqslant 0$, the $(k+1) \times (k+1)$ matrix

$$[c_{m,n}]_{0\leqslant m,n\leqslant k} \tag{1.3a}$$

is positive definite.

This last condition can usefully be expressed in operator terms. For any $m, n \geqslant 0$ we can define the operator P_{mn} on $L^2(\mathbb{R})$ by the formula

$$P_{mn}\psi = \langle h_n, \psi\rangle\, h_m \qquad \psi \in L^2(\mathbb{R}). \tag{1.3b}$$

Then condition (1.3a) can be expressed by requiring that the operators

$$\sum_{m,n=0}^{k} c_{mn}\, P_{mn} \qquad k \geqslant 0 \tag{1.3c}$$

are all positive operators on $L^2(\mathbb{R})$.

Key to the formulation of Lahti and Pellonpää are the Fourier coefficients of the (characteristic functions of the) Borel subsets of $\mathbb{T}$. These are defined as follows: for every Borel set $V \in \mathcal{B}(\mathbb{T})$, we write

$$\tilde{V}_k = \frac{1}{2\pi}\int_V \mathrm{e}^{\mathrm{i}k\theta}\,\mathrm{d}\theta \qquad k \in \mathbb{Z}. \tag{1.4}$$

Then the main result of Lahti and Pellonpää can be summarized as follows.

Proposition 1.2. *For any $c \in \mathfrak{k}$ the operator identity*

$$\kappa(c)(V) = \sum_{m,n=0}^{\infty} c_{mn}\tilde{V}_{n-m} P_{mn} \qquad V \in \mathcal{B}(\mathbb{T}) \tag{1.5}$$

converges to a positive operator on $L^2(\mathbb{R})$ for any $V \in \mathcal{B}(\mathbb{T})$, and this formula defines a GPOVM $\kappa(c)$ on $L^2(\mathbb{R})$. Moreover, the map $\kappa : \mathfrak{k} \to \mathcal{GM}$ is a bijection, with inverse

$$\kappa^{-1}(E)_{mn} = \lim_{\varepsilon\to 0}\frac{2\pi}{\varepsilon}\,\langle h_m, E([0,\varepsilon))h_n\rangle. \tag{1.6}$$

Note that the defining conditions (1.2) and (1.3a) for elements of $\mathfrak{k}$ are sufficient to ensure the summability of the series in (1.5).

2. First moments

POVMs on $\mathbb{R}$ have first moments which are symmetric operators, and conversely (see [5]). Since $\mathbb{T}$ is a bounded subset of $\mathbb{R}$, the first moments of POVMs (and in particular GPOVMs) on $\mathbb{T}$ have first moments which are bounded self-adjoint operators. Consequently, it should be possible to recast the theory of GPOVMs entirely in terms of self-adjoint operators.

To observe this, we recall that the first moment A_E of a POVM E on $\mathbb{T}$ is the bounded self-adjoint operator

$$A_E = \frac{1}{2\pi}\int_0^{2\pi} \theta \, \mathrm{d}E(\theta). \tag{2.7}$$

If $E = \kappa(c) \in \mathcal{GM}$ is a GPOVM derived from the sequence $c \in \mathfrak{k}$, it is clear that

$$\begin{aligned}\langle h_m, A_E h_n\rangle &= \frac{1}{2\pi} c_{mn} \int_0^{2\pi} \theta \, \mathrm{e}^{\mathrm{i}(n-m)\theta} \, \mathrm{d}\theta \\ &= \begin{cases} \pi & m = n \\ \dfrac{1}{\mathrm{i}(n-m)} c_{mn} & m \neq n \end{cases}\end{aligned} \tag{2.8}$$

from which we obtain the identity

$$c_{mn} = \delta_{mn} - \mathrm{i}(m-n)\langle h_m, A_E h_n\rangle \qquad m, n \geqslant 0. \tag{2.9}$$

Thus it is clear that the first moment A_E completely determines the sequence $c \in \mathfrak{k}$ (and hence the GPOVM E), and vice versa. This uniqueness result has been obtained by Lahti and Pellonpää [4].

It is interesting to note a direct relationship between the GPOVM E and its first moment A_E other than the basic one indicated in (2.7). The following relationship reflects the gauge invariance of the situation.

Proposition 2.1. *If E is a GPOVM on $L^2(\mathbb{R})$ and A_E is its first moment, then*

$$U_\theta^{-1} A_E U_\theta + \theta I = A_E + 2\pi E[0, \theta) \tag{2.10}$$

for any $0 \leqslant \theta < 2\pi$.

Proof. Using elementary integration, it is clear that

$$\begin{aligned}\langle h_m, U_\theta^{-1} A_E U_\theta h_n\rangle &= \langle U_\theta h_m, A_E U_\theta h_n\rangle \\ &= \mathrm{e}^{\mathrm{i}(n-m)\theta} \langle h_m, A_E h_n\rangle \\ &= \frac{1}{2\pi} c_{mn} \mathrm{e}^{\mathrm{i}(n-m)\theta} \int_0^{2\pi} \beta \, \mathrm{e}^{\mathrm{i}(n-m)\beta} \, \mathrm{d}\beta \\ &= \frac{1}{2\pi} c_{mn} \int_0^{2\pi} \beta \, \mathrm{e}^{\mathrm{i}(n-m)(\theta+\beta)} \, \mathrm{d}\beta \\ &= \langle h_m, A_E h_n\rangle - \theta\delta_{mn} + 2\pi\langle h_m, E[0,\theta) h_n\rangle\end{aligned}$$

for any $m, n \geqslant 0$, as required. □

Since the first moment A_E of a GPOVM E determines that GPOVM completely, it is evident that there should be a characterization of those bounded self-adjoint operators on $L^2(\mathbb{R})$ which are the first moments of GPOVMs. It is instructive to provide this. Comparison of the positivity condition (1.3a) for the defining sequence $(c_{mn})_{m,n\geqslant 0}$ of a GPOVM and the relationship (2.9) between that sequence and the matrix coefficients of the first moment shows that the first moment operator A_E for a GPOVM satisfies the inequality

$$\mathrm{i}[\langle Nf, A_E f\rangle - \langle f, A_E N f\rangle] \leqslant \|f\|^2 \qquad f \in \mathcal{S}(\mathbb{R}) \tag{2.11}$$

and we can use this observation to arrive at the following result.

Proposition 2.2. *If A is a bounded self-adjoint operator on $L^2(\mathbb{R})$, then there exists a GPOVM E on $L^2(\mathbb{R})$ and a bounded (measurable) real-valued function G on $\mathbb{N} \cup \{0\}$ such that*

$$A = A_E + G(\overline{N}) \tag{2.12a}$$

if and only if A satisfies the inequality

$$\mathrm{i}[\langle Nf, Af\rangle - \langle f, ANf\rangle] \leqslant \|f\|^2 \qquad f \in \mathcal{S}(\mathbb{R}). \tag{2.12b}$$

In addition to this, $G = 0$, and so $A = A_E$, if and only if $\langle h_n, Ah_n\rangle = \pi$ for all $n \in \mathbb{N} \cup \{0\}$.

Proof. Firstly, it is clear from the above discussion that the first moment A_E of a GPOVM satisfies equation (2.12*b*). Moreover, since the commutator of any function $G(\overline{N})$ of the number operator $\overline{N}$ with itself vanishes, any bounded self-adjoint operator A on $L^2(\mathbb{R})$ of the form indicated in equation (2.12*a*) also satisfies equation (2.12*b*).

Conversely, suppose that A is a bounded self-adjoint operator on $L^2(\mathbb{R})$ which satisfies equation (2.12*a*). Then it is clear that defining a double sequence $(c_{mn})_{m,n\geqslant 0}$ by the formula

$$c_{mn} = \delta_{mn} - \mathrm{i}(m-n)\langle h_m, Ah_n\rangle \qquad m, n \geqslant 0$$

yields an element $c \in \mathfrak{k}$, which can be used to define a GPOVM $E = \kappa(c)$. It is then clear from equation (2.9) that the off-diagonal matrix coefficients for A and A_E coincide, and hence that $A - A_E$ is a diagonal operator on $L^2(\mathbb{R})$ (with respect to the Hermite–Gauss functions), and hence is of the form $G(\overline{N})$ for some bounded (measurable) real-valued function G on $\mathbb{N} \cup \{0\}$. This completes the proof. □

It is worth noting that the relationship between the GPOVM E and its first moment A_E indicated by equation (2.10) is complete in that it permits us to reconstruct E solely from its first moment. To be specific, suppose that A is a bounded self-adjoint operator on $L^2(\mathbb{R})$ which satisfies equation (2.12*b*). Equation (2.10) indicates that we should consider the operators

$$\mathcal{E}(\theta) = \frac{1}{2\pi}[U_\theta^{-1}\, A\, U_\theta - A + \theta I] \qquad \theta \in [0, 2\pi].$$

The positivity condition (2.12*b*) can be used to show that any such operator $\mathcal{E}(\theta)$ is positive, and moreover that the map $\mathcal{E}$ from $[0, 2\pi]$ to the positive bounded operators on $L^2(\mathbb{R})$ is increasing, with $\mathcal{E}(0) = 0$ and $\mathcal{E}(2\pi) = I$. Standard measure-theoretic gymnastics enable us to reconstruct a POVM E on $L^2(\mathbb{R})$ such that $\mathcal{E}(\theta) = E[0, \theta)$ for any $\theta \in \mathbb{T}$, and the nature of equation (2.10) ensures that the POVM constructed by this process is indeed gauge covariant. Thus the POVM E so obtained is a GPOVM, and its first moment A_E and the original operator A are related by equation (2.12*a*).

3. Consequences

Were we to play fast and loose[3] with our notation, we might observe that equation (2.12*b*) could be written in the operator form

$$\mathrm{i}[\overline{N}, A] \leqslant I. \tag{3.13}$$

It is well known in quantum optics that there are no self-adjoint operators which satisfy the identity $\mathrm{i}[N, A] = I$ in any sense (weak or otherwise) on a domain in $L^2(\mathbb{R})$ which includes the linear span of the Hermite–Gauss functions (the no–go theorem [2]). There are, however, many self-adjoint bounded operators which satisfy this weaker inequality (3.13). Since any

[3] To be able to work with this inequality rigorously, we need to know whether A maps $\mathcal{S}(\mathbb{R})$ into the domain of $\overline{N}$—a question we choose not to address. However, the inequality is suggestive, and a convenient shorthand.

function of $\overline{N}$ can be added to A without affecting this inequality, there is no problem in finding bounded self-adjoint operators on $L^2(\mathbb{R})$ which satisfy this inequality, together with the condition

$$\langle h_n, A h_n \rangle = \pi \qquad n \geqslant 0$$

and hence which are first moments of elements of $\mathcal{GM}$. A number of examples are given by Lahti and Pellonpää [3, 4], and we want to use one such example to show that, starting from E and forming A_E, the spectral decomposition of A_E can be rather different in character from what E would have one suspect.

Consider the simple example of the element of $\mathcal{GM}$ given in terms of a pair of distinct non-negative integers s, t and a complex number z of modulus less than or equal to unity, obtained by setting $c_{st} = z$, $c_{ts} = \overline{z}$, with all other off-diagonal elements being zero. This leads to the formula

$$E(V) = \frac{1}{2\pi}\int_V \mathrm{d}\beta + \frac{z}{2\pi}\int_V \mathrm{e}^{\mathrm{i}(t-s)\beta}\,\mathrm{d}\beta\, P_{st} + \frac{\overline{z}}{2\pi}\int_V \mathrm{e}^{\mathrm{i}(s-t)\beta}\,\mathrm{d}\beta\, P_{ts}$$

for any $V \in \mathcal{B}(\mathbb{T})$. The associated first moment observable A_E can be read off from this:

$$A_E = \pi I + \frac{z}{\mathrm{i}(t-s)} P_{st} + \frac{\overline{z}}{\mathrm{i}(s-t)} P_{ts}$$

and this self-adjoint operator is particularly simple to analyse. Its spectrum consists of the three eigenvalues

$$\pi + \frac{|z|}{t-s}, \qquad \pi - \frac{|z|}{t-s}, \qquad \pi$$

the first two of which are nondegenerate, with associated eigenvectors

$$h_s + \mathrm{i}\mathrm{e}^{-\mathrm{i}\varphi} h_t, \qquad h_s - \mathrm{i}\mathrm{e}^{-\mathrm{i}\varphi} h_t$$

where $z = |z|\mathrm{e}^{\mathrm{i}\varphi}$. Thus the three eigenvalues have the associated spectral projections

$$\begin{aligned} P_+ &= \tfrac{1}{2}[P_{ss} + P_{tt} + \mathrm{i}(\mathrm{e}^{-\mathrm{i}\varphi} P_{ts} - \mathrm{e}^{\mathrm{i}\varphi} P_{st})] \\ P_- &= \tfrac{1}{2}[P_{ss} + P_{tt} - \mathrm{i}(\mathrm{e}^{-\mathrm{i}\varphi} P_{ts} - \mathrm{e}^{\mathrm{i}\varphi} P_{st})] \\ P_\pi &= I - P_{ss} - P_{tt} \end{aligned}$$

so that while we have

$$A_E = \pi P_\pi + \left(\pi + \frac{|z|}{t-s}\right) P_+ + \left(\pi - \frac{|z|}{t-s}\right) P_-$$

we see that, in general, $E(V)$ is not a linear combination of P_+, P_- and P_π, and certainly $E(V)$ is not the spectral projection for V associated with A_E, which is simply the sum of as many of the above three projections whose associated eigenvalues belong to V.

The standard interpretation of quantum theory now tells us that A_E is an observable (in the sense of Dirac and von Neumann), independent of the means by which we obtained it. Because of its simple spectrum, it is possible (in principle) to arrange a measurement of it which would result in the precise measurement of its eigenvalues. This requires no more than an arrangement capable of distinguishing amongst the three eigenvalues of A_E. Subject to the eigenvalue obtained (registered), such an apparatus prepares the corresponding eigenstate.

This statement does not refer to E, and is arrangement independent, subject to the resolution condition. But as A_E is the first moment of E, and as E represents a measurement arrangement, we must connect the two in this regard.

A POVM, generally, represents an experimental arrangement which, for a given state, results in values for the expectations of the $E(V)$ for all Borel sets V (in the domain of definition of E). By piecing these positive numbers together to form the first moment, A_E will result. How effectively this can be done depends, firstly, on how accurately E is being measured. This is not a trivial observation, since the spectrum of E is continuous, and so subject to at least some minimum limitations in its measurement. Secondly, there will be an error in using this data to compute the first moment. However, in the usual spirit of physics, we may consider A_E as an idealized limit, rather than a directly measured quantity.

(The curious phenomenon of a POVM with a continuous spectrum (here $[0, 2\pi)$) with a first moment with a discrete spectrum (here three eigenvalues) is not new.)

But there is a third point to consider, for there are output states associated with the measurements of the $E(V)$, and these are not the same as the eigenstates of A_E. It does not seem sensible to *prepare* an A_E eigenstate by an E-arrangement.

An interesting point to consider is that if we were faced with A_E in its eigendecomposition form, there would be no angle to consider. Only by looking amongst the POVMs which have it as their first moment would E and its angle appear. So while E is a perfectly good observable measure with gauge angle covariance, it has a first moment which does not. Even if you believe in considering POVMs as the primary notion of observable in quantum mechanics, the old notion of observable must still remain (since $\{P_-, P_+, P_\pi\}$ is a special sort of POVM), and so A_E must be considered as part of the quantal information. But then it has to be asked: what is the significance of the covariance angle when its first moment has no such covariance? Physically, how is the physical content encoded in A_E related to the covariance angle? Could we ascribe a phase to A_E? If so, if there were another GPOVM whose first moment was A_E (such a thing is possible), how would the phases be related physically?

One class of answer might be the one indicated above: considering that POVMs should be the primary notion of observable in quantum mechanics. This follows the tradition begun by von Neumann and Birkhoff, although they considered the rather more restricted structure of the lattice of projection operators as propositions, or questions. Then the fact that A_E is so different in character from E might not be anything to worry about, and the fact that a covariant POVM like E and a non-covariant POVM like $\{P_-, P_+, P_\pi\}$ have the same first moment is of no consequence. This is not our belief.

For these reasons, the authors are very chary about using the term phase observable for GPOVMs, though certainly not everyone agrees with us. The mathematics of GPOVMs found in this paper, and in [3], is clear, and, as long as the terms *phase* and *observable* are used in a precisely mathematically defined way, there can be no objections. We must all be wary of confusing the name with the thing. We have included a very few references discussing POVMs and quantum mechanics, [6–12].

Our reluctance to identify GPOVMs as phase observables is increased by the following example of an operator which, we believe, has a strong case to be considered as a phase observable, and yet which has no GPOVM decomposition. The operator in question is the Weyl quantization, $\boldsymbol{\Delta}[\varphi]$, of the angle function in phase space, [1,2]. In view of its construction, it necessarily has a phase space angle as its classical limit. But what classical limit? We have shown in [13] that it is the phase of the light in the coherent phase in the modified Dick laser model, and in this model the macroscopic description of light is classical. (We note that none of the other candidate phase operators that we know of has this property. They are all functions of both the phase and intensity of the light in this model, though, interestingly, they converge to the phase as the intensity approaches infinity.) This phenomena is a sort of reverse to that exhibited by E and A_E.

To prove this assertion we need results concerning the commutator of $\boldsymbol{\Delta}[\varphi]$ and the number operator N. This cannot be defined as a simple operator, but can be represented as a sesquilinear form $\tau_{N,\boldsymbol{\Delta}[\varphi]}$ on $\mathcal{S}(\mathbb{R})$. Details may be found in [1,2]. The result turns out to be that

$$\begin{aligned}\mathrm{i}[\langle Nf, \boldsymbol{\Delta}[\varphi]f\rangle - \langle f, \boldsymbol{\Delta}[\varphi]Nf\rangle] &= \mathrm{i}\tau_{N,\boldsymbol{\Delta}[\varphi]}[f, f] \\ &= \|f\|^2 + \tfrac{1}{2}([Uf'](\overline{f}) - [Uf](\overline{f}')) \qquad (3.14a)\end{aligned}$$

for any $f \in \mathcal{S}(\mathbb{R})$, where $U \in \mathcal{L}(\mathcal{S}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$ is defined by the formula

$$[Ug](f) = \lim_{L\to\infty} \int_{\mathbb{R}} g_{I(L)}(x) f(x) g(-x)\,\mathrm{d}x \qquad (3.14b)$$

for $f, g \in \mathcal{S}(\mathbb{R})$, $g_{I(L)}$ being the cut-off factor

$$g_{I(L)} = \begin{cases} x^{-1} & L^{-1} < |x| < L \\ 0 & \text{otherwise.} \end{cases} \qquad (3.14c)$$

Direct calculation, with $f(x) = \mathrm{e}^{-\frac{1}{2}x^2}$ yields

$$[Uf'](\overline{f}) = -[Uf](\overline{f}') = \int_{\mathbb{R}} \mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$$

so that

$$\mathrm{i}[\langle Nf, \boldsymbol{\Delta}[\varphi]f\rangle - \langle f, \boldsymbol{\Delta}[\varphi]Nf\rangle] > \|f\|^2$$

in this case. Thus $\boldsymbol{\Delta}[\varphi]$ is not the first moment of a GPOVM.

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